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Vindication of mode-coupled descriptions of multiple-scale water wave fields

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Herein we show that the modal description of deep-water waves on the sea surface (Watson & West 1975) is independent of any reference surface around which expansions of the velocity potential and the surface velocity are done. We demonstrate by direct construction that the interaction between long and short waves does not lead to divergent terms in the equations of motion when this formalism is used.

1. Introduction

In this paper we attempt to lay to rest the criticism that mode-coupling theories are incapable of describing the interaction between surface water waves widely separated in scale. The criticism arises from the observation that low-order approximations to the velocity potential diverge as a product of the long-wave amplitude and the short-wave wavenumber (see e.g. O. M. Phillips 1978, unpublished report). This flaw results in an apparent divergence in the dynamic equations describing the interaction of two waves having widely separated wavelengths. This apparently divergent approximation results from an expansion of the velocity potential about the $z = 0$ surface of the water. Holliday (1977) considered a variant of this problem in which the average shift in frequency of a gravity–capillary wave produced by an ambient spectrum of such waves was used as a measure of convergence of the perturbation series. He calculated this shift on both the surface $z = 0$ and the free surface $z = \zeta$, and found that the perturbation series converges more rapidly on the latter surface. His analysis established that the lowest-order perturbation results of Hasselmann (1962) ($z = 0$) were not usable, whereas those of Watson & West (1975) ($z = \zeta$) should be used for analysing nonlinear interactions in a spectrum of surface waves.

Holliday's arguments have not found universal acceptance, in part because their applicability to interacting surface waves widely separated in scale is unclear. Herein we present a more direct proof of the convergence of the perturbation series describing the interaction between long waves and short waves. To better understand the formal argument let us first review the essential features of the criticism.

Consider the vertical height of the sea surface $\zeta(\boldsymbol{x}, t)$ and the velocity potential $\phi(\boldsymbol{x}, z, t)$ which satisfies Laplace's equation $\nabla^2\phi = 0$ in the ocean interior. The Fourier expansion of the free surface is given by

$$\zeta(\boldsymbol{x}, t) = \tfrac{1}{2}\sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\zeta_{\boldsymbol{k}}(t) + \mathrm{c.c.} \tag{1.1}$$

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and that of the velocity potential satisfying the boundary condition $\phi = 0$ at $z = -\infty$ by

$$\phi(\boldsymbol{x}, z, t) = \tfrac{1}{2} \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \mathrm{e}^{kz} \phi_{\boldsymbol{k}}(t) + \text{c.c.}, \tag{1.2}$$

and c.c. denotes the complex conjugate of the first term. At the free surface $z = \zeta(\boldsymbol{x}, t)$ the velocity potential becomes an exponential function of ζ, but in the equations of motion one uses the Taylor series expansion about the $z = 0$ reference surface:

$$\phi[\boldsymbol{x}, z = \zeta(\boldsymbol{x}, t), t] = \tfrac{1}{2} \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \phi_{\boldsymbol{k}}(t) \sum_{n=0}^{\infty} \frac{(k\zeta)^n}{n!} + \text{c.c.} \tag{1.3}$$

However the surface displacement $\zeta(\boldsymbol{x}, t)$ itself consists of many scales as is evidenced by the wave vector series (1.1), so that one cannot guarantee the convergence of (1.3) in general.

For concreteness let us consider the superposition of two linear one-dimensional waves on which to test the convergence of the expansion (1.3). The surface height is given by

$$\zeta(x, t) = a_1 \cos(k_1 x - \omega_1 t) + a_2 \cos(k_2 x - \omega_2 t) \tag{1.4}$$

and we choose $a_1 = 10$ cm, $k_1 = 0.01\ \text{cm}^{-1}$, $a_2 = 1$ cm and $k_2 = 0.1\ \text{cm}^{-1}$ so that the slope of both waves (ka) is 0.1. For the argument here we take the maximum surface height to be given by the amplitude of the longer wave, $\bar{\zeta} \sim 10$ cm. Only a modest numerical error is introduced by this approximation and it does not affect the general conclusion of our argument. In the sum on wavenumber in (1.3) we have the two values k_1 and k_2, so that the velocity potential becomes

$$\phi(x, \bar{\zeta}, t) \approx \phi_1(t) \sin(k_1 x - \omega_1 t) \sum_{n=0}^{\infty} \frac{(k_1 \bar{\zeta})^n}{n!} + \phi_2(t) \sin(k_2 x - \omega_2 t) \sum_{n=0}^{\infty} \frac{(k_2 \bar{\zeta})^n}{n!}. \tag{1.5}$$

In the first term we have $k_1 \bar{\zeta} = 0.1$ so the series converges rapidly. However $k_2 \bar{\zeta} = 1$ so the second series converges very slowly. Thus, the expansion of the shorter wave about the $z = 0$ surface leads to a divergent result unless all the terms in the series are kept. This is not entirely unexpected since the shorter wave rides atop the longer one and is displaced quite far from the $z = 0$ plane. An expansion in the vicinity of the reference plane for a wave whose wavelength is shorter than the displacement from the plane would naturally lead to non-physical results.

Based on the above argument it is often asserted that the modal expansions are only useful for very narrow spectra, since only a few terms in the series for the velocity potential can be retained in the equations of motion for the sea surface. The spectrum cannot be too broad in order that the Taylor expansion about the reference surface converge rapidly. Herein we show that even though the expansion of the velocity potential about a reference surface diverges when truncated at finite order, the equations of motion for the free surface are well behaved. We demonstrate that in the formalism of Watson & West (1975, hereinafter referred to as WW) the formally divergent terms cancel against each other in the equations of motion.

In §2 we briefly sketch the WW formalism and show that the vertical velocity and the velocity potential at the free surface are independent of any reference surface. It is shown that this independence implies that the approximate equations of motion are expansions in the surface slope and not in the surface height (see e.g. WW; West *et al.* 1987). In §3 we apply the result of this formalism to the interaction of two waves separated in scale and show explicitly that the divergent terms cancel in the equations of motion. In §4 we discuss the original mode-coupled equations of Hasselmann (1962, 1963*a*, *b*) and argue that they too may be freed of the multiple-scale criticism if enough terms are taken.

2. Watson–West formalism

The equations of motion for an irrotational, incompressible, inviscid water surface are given by Bernoulli's equation and the kinematic boundary condition:

$$\frac{\partial \phi(\boldsymbol{x},z,t)}{\partial t}+\tfrac{1}{2}\nabla\phi(\boldsymbol{x},z,t)\cdot\nabla\phi(\boldsymbol{x},z,t)+gz=0;\quad z=\zeta(\boldsymbol{x},t), \tag{2.1a}$$

$$\frac{\partial \phi(\boldsymbol{x},t)}{\partial t}+\nabla\phi(\boldsymbol{x},z,t)\cdot\nabla\zeta(\boldsymbol{x},t)=\frac{\partial \phi(\boldsymbol{x},z,t)}{\partial z};\quad z=\zeta(\boldsymbol{x},t). \tag{2.1b}$$

In the interior of the incompressible fluid $\nabla\cdot\boldsymbol{v}=0$ so that the velocity potential satisfies Laplace's equation $\nabla^2\phi=0$. Using this latter condition and the velocity potential defined on the free surface $\phi_s(\boldsymbol{x},t)=\phi[\boldsymbol{x},z=\zeta(\boldsymbol{x},t),t]$ these equations become (WW)

$$\frac{\partial \phi_s}{\partial t}(\boldsymbol{x},t)+\tfrac{1}{2}\nabla_s\phi_s(\boldsymbol{x},t)\cdot\nabla_s\phi_s(\boldsymbol{x},t)+g\zeta(\boldsymbol{x},t)=\tfrac{1}{2}[1+\nabla_s\zeta(\boldsymbol{x},t)\cdot\nabla_s\zeta(\boldsymbol{x},t)]\,W(\boldsymbol{x},t)^2, \tag{2.2a}$$

$$\frac{\partial}{\partial t}\zeta(\boldsymbol{x},t)+\nabla_s\phi_s(\boldsymbol{x},t)\cdot\nabla_s\zeta(\boldsymbol{x},t)=[1+\nabla_s\zeta(\boldsymbol{x},t)\cdot\nabla_s\zeta(\boldsymbol{x},t)]\,W(\boldsymbol{x},t), \tag{2.2b}$$

where the vertical velocity is given by

$$W(\boldsymbol{x},t)=\frac{\partial}{\partial z}\phi(\boldsymbol{x},z,t)|_{z=\zeta(\boldsymbol{x},t)} \tag{2.3}$$

and ∇_s is the horizontal gradient operator. To solve these equations we express W as an explicit function of $\phi_s(\boldsymbol{x},t)$ and $\zeta(\boldsymbol{x},t)$, in which case (2.2) constitutes a set of nonlinear equations in these two field variables.

WW obtain a formal solution for the vertical velocity W using Laplace's equation and by expanding about a reference surface ζ_0,

$$\phi[\boldsymbol{x},\zeta(\boldsymbol{x},t),t]=\sum_{n=0}^{\infty}\frac{[(\zeta(\boldsymbol{x},t)-\zeta_0)^n]}{n!}\kappa^n\phi(\boldsymbol{x},\zeta_0,t), \tag{2.4}$$

$$W(\boldsymbol{x},t)=\sum_{n=0}^{\infty}\frac{[\zeta(\boldsymbol{x},t)-\zeta_0]^n}{n!}\kappa^{n+1}\phi(\boldsymbol{x},\zeta_0,t), \tag{2.5}$$

where κ is an operator which multiplies any one- or two-dimensional Fourier coefficient of $\phi(\boldsymbol{x},\zeta_0,t)$ by the magnitude of the wave vector $\boldsymbol{k}$. The series (2.4) and (2.5) can be rearranged because ζ_0 commutes with κ to yield

$$\phi_s(\boldsymbol{x},t)=\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\kappa^n\,e^{-\zeta_0\kappa}\,\phi(\boldsymbol{x},\zeta_0,t), \tag{2.6}$$

$$W(\boldsymbol{x},t)=\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\kappa^{n+1}\,e^{-\zeta_0\kappa}\,\phi(\boldsymbol{x},\zeta_0,t). \tag{2.7}$$

The product $e^{-\zeta_0\kappa}\phi(\boldsymbol{x},\zeta_0,t)$ does not change for different choices of ζ_0. Even more explicitly if we write

$$\phi_s(\boldsymbol{x},t) = \mathrm{O}(\boldsymbol{x},t)\, e^{-\zeta_0\kappa}\phi(\boldsymbol{x},\zeta_0,t), \tag{2.8}$$

$$W(\boldsymbol{x},t) = Q(\boldsymbol{x},t)\, e^{-\zeta_0\kappa}\phi(\boldsymbol{x},\zeta_0,t), \tag{2.9}$$

then if the operator $\mathrm{O}(\boldsymbol{x},t)$ has an inverse $\mathrm{O}^{-1}(\boldsymbol{x},t)$ we can write

$$W(\boldsymbol{x},t) = Q(\boldsymbol{x},t)\,\mathrm{O}^{-1}(\boldsymbol{x},t)\,\phi_s(\boldsymbol{x},t) \tag{2.10}$$

independent of the reference surface (see West 1981). The expression for the vertical velocity (2.10) is therefore a series in the surface slope in that only differences in the wave height enter, i.e. a shift of ζ by ζ_0 does not modify W. From this we can conclude that the equations of motion are independent of ζ_0.

3. Cancellation of divergent terms

It may not be obvious how the formal result for the vertical velocity (2.10) influences the two-scale example worked out in §1, and, in particular, how this result changes the conclusions arrived at there. Consider again the case of two waves with very different wavelengths and amplitudes [cf. (1.4)] for which the velocity potential in one dimension is written

$$\phi_s(x) = \frac{\omega_1}{k_1}a_1 \sin(k_1 x) + \frac{\omega_2}{k_2}a_2\sin(k_2 x), \tag{3.1}$$

where for the moment we suppress the time index for the sake of clarity. The formal expression for W, (2.10), can be written

$$W = \sum_{n=0}^{\infty} W_n \tag{3.2}$$

and evaluating the product $Q\mathrm{O}^{-1}$ (see West 1981) the first two terms in (3.2) can be written

$$W_0 = \kappa\phi_s, \tag{3.3}$$

$$W_1 = (\zeta\kappa^2 - \kappa\zeta\kappa)\,\phi_s, \tag{3.4}$$

$$\vdots$$

Note that W_1 has the form of a commutator opening on $\kappa\phi_s$, i.e. $[\zeta,\kappa] = \zeta\kappa - \kappa\zeta$.

The first term (3.3) is well behaved. The second term is given explicitly by

$$\begin{aligned} W_1 = {} & [a_1\cos(k_1 x) + a_2\cos(k_2 x)]\,\kappa^2\left[\frac{\omega_1}{k_1}a_1\sin(k_1 x) + \frac{\omega_2}{k_2}a_2\sin(k_2 x)\right] \\ & - \kappa[a_1\cos(k_1 x) + a_2\cos(k_2 x)]\,\kappa\left[\frac{\omega_1}{k_1}a_1\sin(k_1 x) + \frac{\omega_2}{k_2}a_2\sin(k_2 x)\right]. \end{aligned} \tag{3.5}$$

A typical cross-term of (3.5) is given by

$$I = a_1 a_2 \frac{\omega_2}{k_2}[\cos(k_1 x)\,\kappa^2\sin(k_2 x) - \kappa\cos(k_1 x)\,\kappa\sin(k_2 x)], \tag{3.6}$$

where using the identity

$$\begin{aligned} \kappa\cos(k_1 x)\,\kappa\sin(k_2 x) = {} & \tfrac{1}{2}[|k_1+k_2| + |k_1-k_2|]\,k_2\sin(k_2 x)\cos(k_1 x) \\ & + \tfrac{1}{2}(|k_1+k_2| - |k_1-k_2|)\,k_2\sin(k_1 x)\cos(k_2 x) \end{aligned}$$

one obtains

$$I = a_1 a_2 \frac{\omega_2}{k_2} \{[k_2^2 - \tfrac{1}{2}k_2(|k_1+k_2|+|k_1-k_2|)] \sin(k_2 x) \cos(k_1 x) - \tfrac{1}{2}k_2(|k_1+k_2|-|k_1-k_2|) \sin(k_1 x) \cos(k_2 x)\}. \quad (3.7)$$

Thus, if as in §1 $a_1 \gg a_2$ and $k_1 \ll k_2$, (3.7) reduces to

$$I \approx -a_1 a_2 \omega_2 k_1 \sin(k_1 x) \cos(k_2 x), \quad (3.8)$$

demonstrating that the cross-term dependence on $k_2 a_1$ cancels. Since this term is typical, we see that all terms involving $k_2 a_1$ vanish from W_1. The WW formalism therefore combines the partially divergent series for the velocity potential and that for the vertical velocity in such a way as to yield a convergent series for W directly from ζ and ϕ_S. In our two-scale example this results in

$$W_1 \approx -\frac{\omega_1}{k_1}(k_1 a_1)^2 \sin(k_1 x) \cos(k_1 x) - \frac{\omega_2}{k_2}(k_2 a_2)^2 \sin(k_2 x) \cos(k_2 x) - \frac{\omega_2}{k_2}(k_1 a_1)(k_2 a_2) \sin(k_2 x) \cos(k_1 x), \quad (3.9)$$

which is an explicit expansion in the wave slopes $k_1 a_1$ and $k_2 a_2$, and is well behaved for $k_1 a_1 < 1$ and $k_2 a_2 < 1$. This cancellation is a consequence of the commutation operator $[\zeta, \kappa]$ acting on $\kappa\phi_S$ on the right-hand side of (3.4). The higher-order factors in the series expansion for W can similarly be expressed in terms of commutators, thereby ensuring the cancellation of the diverging cross-terms discussed above.

The formal equations of motion can still be written as a set of nonlinear mode-coupled rate equations (see e.g. Hasselmann 1962; WW; West *et al.* 1987). What distinguishes one set of equations from another is the choice of coupling coefficients determining the strength of the interaction of one wave with another. In the above example the W_1 term leads to cubic nonlinearities in (2.2), with coupling coefficients that are at least quadratic in the slopes of the two interacting waves. This is where most mode-coupling theories leave the analysis, with the notable exception of West *et al.* (1987) who extend the numerical integration of the equations of motion beyond third order in the mode amplitudes.

4. Conclusions

We have shown that even though the expansions of the velocity potential and the vertical velocity about a reference surface may be formally divergent, these series can be reordered in such a way that no such divergence in fact occurs at a given order in the appropriate variables in the equations of motion. This reordering is given explicitly by the WW formalism and shows that the dynamic equations on the free surface are expressible as an expansion in the surface slope. We worked out the example of a long wave interacting with a short wave and demonstrated that to second order in the vertical velocity the would-be divergent terms cancel. (These terms are third order in the equations of motion). The third-order terms in W can also be worked out explicitly, but the algebra becomes onerous. (These terms are fourth order in the equations of motion.) We rely on the formal proof in §2 that the velocity potential ϕ_S is independent of the reference surface to establish this result in general and note that the numerical integration of the dynamic equations support this result (see West *et al.* 1987).

It is worth emphasizing that in the WW formalism each term in the perturbation series for the vertical velocity can be expressed in terms of commutators. The formal structure of the commutators ensures the cancellation of the divergent terms discussed in §3.

We note that the original mode-coupled equations constructed by Hasselmann (1962), were quite different from those obtained from (2.2). This difference arises in part because Hasselmann makes a double expansion, one in the velocity potential as we do, but also a second expansion in the surface height

$$\phi(\boldsymbol{x}, \zeta_0, t) = \sum_n \phi_n(\boldsymbol{x}, \zeta_0, t), \tag{4.1}$$

$$\zeta(\boldsymbol{x}, t) = \sum_n \zeta_n(\boldsymbol{x}, t), \tag{4.2}$$

where again the index n gives the order of the dependence of ϕ_n and ζ_n in the perturbation solution by successive approximation. Without presenting any of the details we state that the Hasselmann perturbation equations can be obtained from those of WW by introducing the expansion in the height (4.2) into the latter (cf. the Appendix). These equations can subsequently be shown to be independent of the reference height. We have been able to do this explicitly up to third order in the equation of motion, but again the algebra becomes quite difficult at fourth-order. Note that truncating the Hasselmann expansion at a given order destroys the commutator form of the series, whereas the WW expansion is expressible in terms of commutators at each order.

As mentioned, Holliday (1977) determined that the lowest-order terms in the Hasselmann expansion about $z = 0$ produces results that 'by themselves are meaningless'. It is possible however to regroup the terms in that analysis to yield those of the WW perturbation series, i.e. to do a resummation of terms so as to define the mode amplitudes on the free surface rather than on the $z = 0$ surface. In the Appendix we show explicitly for the first time how the two series are related, from which the reader can determine the result of truncating the series at a given order.

Thus it would appear that the major criticism against the use of mode-coupled equations to describe the evolution of surface water waves does not hold water.

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Appendix. Hasselmann method

The Hasselmann equation can be obtained by starting from the Bernoulli equations

$$\left.\begin{aligned} &\frac{\partial \phi}{\partial t} + \tfrac{1}{2}(\boldsymbol{\nabla}_s \phi)^2 + \tfrac{1}{2}(\kappa\phi)^2 + g(\zeta - \zeta_0) = 0 \\ &\frac{\partial h}{\partial t} + \boldsymbol{\nabla}_s \phi \cdot \boldsymbol{\nabla}_s h - \kappa\phi = 0 \end{aligned}\right\} \quad \text{at } z = \zeta, \tag{A 1}$$

with ζ_0 the height of an arbitrarily chosen reference surface. Hasselmann makes the double expansion

$$\left.\begin{aligned}\phi_s(\boldsymbol{x},t) &= \sum_m \frac{(z-\zeta_0)^m \kappa^m}{m!}\phi(\boldsymbol{x},\zeta_0,t) \quad \text{at } z=\zeta,\\ \phi(\boldsymbol{x},\zeta_0,t) &= \sum_n \phi_n(\boldsymbol{x},\zeta_0,t),\\ \zeta(\boldsymbol{x},t) &= \sum_n \zeta_n(\boldsymbol{x},t),\end{aligned}\right\} \qquad \text{(A 2)}$$

with the index n giving the order of ϕ_n and ζ_n in the perturbation solution by successive approximation. Introducing (A 2) into (A 1) gives the sequence of equations

$$\left.\begin{aligned}&\frac{\partial\phi_1}{\partial t}+g(\zeta_1-\zeta_0)=0,\\ &\frac{\partial\phi_2}{\partial t}+(\zeta_1-\zeta_0)\,\kappa\frac{\partial}{\partial t}\phi_1+\tfrac{1}{2}(\boldsymbol{\nabla}_s\phi_1)^2+\tfrac{1}{2}(\kappa\phi_1)^2+g\zeta_2=0,\\ &\frac{\partial\phi_3}{\partial t}+\zeta_2\kappa\frac{\partial\phi_1}{\partial t}+(\zeta_1-\zeta_0)\,\kappa\frac{\partial\phi_2}{\partial t}+\tfrac{1}{2}(\zeta_1-\zeta_0)^2\kappa^2]\frac{\partial\phi_1}{\partial t}\\ &\quad+\boldsymbol{\nabla}_s\phi_1\cdot\boldsymbol{\nabla}_s\phi_2+(\kappa\phi_1)(\kappa\phi_2)+g\zeta_3\\ &\quad+\boldsymbol{\nabla}_s\phi_1\cdot(\zeta_1-\zeta_0)\,\kappa\boldsymbol{\nabla}_s\phi_1+(\kappa\phi_1)(\zeta_1-\zeta_0)\,\kappa^2\phi_1=0,\end{aligned}\right\} \qquad \text{(A 3)}$$

and

$$\left.\begin{aligned}&\frac{\partial\zeta_1}{\partial t}-\kappa\phi_1=0,\\ &\frac{\partial\zeta_2}{\partial t}+\boldsymbol{\nabla}_s\phi_1\cdot\boldsymbol{\nabla}_s\zeta_1-\kappa\phi_2-(\zeta_1-\zeta_0)\,\kappa^2\phi_1=0,\\ &\frac{\partial\zeta_3}{\partial t}+\boldsymbol{\nabla}_s\phi_1\cdot\boldsymbol{\nabla}_s\zeta_2+\boldsymbol{\nabla}_s\phi_2\cdot\boldsymbol{\nabla}_s h_1+(\zeta_1-\zeta_0)\,\kappa\boldsymbol{\nabla}_s\phi_1\cdot\boldsymbol{\nabla}_s\zeta_1-\kappa\phi_3\\ &\qquad-\zeta_2\kappa^2\phi_1-(\zeta_1-\zeta_0)\,\kappa^2\phi_2-\tfrac{1}{2}[(\zeta_1-\zeta_0)^2]\,\kappa^3\phi_1=0.\end{aligned}\right\} \qquad \text{(A 4)}$$

These equations apparently depend on ζ_0. This dependence can be removed by making the replacement

$$\left.\begin{aligned}\tilde{\phi}_1 &= \phi_1 \equiv (\mathrm{e}^{-\kappa\zeta_0}\phi)_1,\\ \tilde{\phi}_2 &= \phi_2-\zeta_0\kappa\phi_1 \equiv (\mathrm{e}^{-\kappa\zeta_0}\phi)_2,\\ \tilde{\phi}_3 &= \phi_3-\zeta_0\kappa\phi_2+\zeta_0^2\tfrac{1}{2}\kappa^2\phi_1 \equiv (\mathrm{e}^{-\kappa\zeta_0}\phi)_3.\end{aligned}\right\} \qquad \text{(A 5)}$$

Substitution of (A 5) into (A 3) and (A 4) shows that all terms in ζ_0 are removed, so that the time evolution of ζ is independent of ζ_0.

The Hasselmann equations can also be derived from the WW equations. The WW time-evolution equations, written as a perturbation series in $\phi_n(\boldsymbol{x}, \zeta, t)$ and ζ_n are, to third order,

$$\left.\begin{aligned}
\frac{\partial \phi_1}{\partial t} &= -g\zeta_1, \\
\frac{\partial \phi_2}{\partial t} &= -g\zeta_2 - \tfrac{1}{2}(\boldsymbol{\nabla}_s \phi_1)^2 (\kappa\phi_1)^2, \\
\frac{\partial \phi_3}{\partial t} &= -g\zeta_3 - \boldsymbol{\nabla}_s \phi_1 \cdot \boldsymbol{\nabla}_s \phi_2 + (\kappa\phi_1)(\kappa\phi_2), \\
\frac{\partial \zeta_1}{\partial t} &= W_1 = \kappa\phi_1, \\
\frac{\partial \zeta_2}{\partial t} &= W_2 - \boldsymbol{\nabla}_s \phi_1 \cdot \boldsymbol{\nabla}_s \zeta_1 \\
&= \kappa\phi_2 + \kappa\zeta_1 \kappa\phi_1 - \boldsymbol{\nabla}_s \phi_1 \cdot \boldsymbol{\nabla}_s \zeta_1, \\
\frac{\partial \zeta_3}{\partial t} &= W_3 - \boldsymbol{\nabla}_s \phi_2 \cdot \boldsymbol{\nabla}_s \zeta_1 - \boldsymbol{\nabla}_s \phi_1 \cdot \boldsymbol{\nabla}_s \zeta_2 + W_1 (\boldsymbol{\nabla}_s \zeta_1)^2 \\
&= \kappa\phi_3 + \zeta\kappa^2\phi_2 + \tfrac{1}{2}\zeta^2\kappa^3\phi_1 - \boldsymbol{\nabla}_s \phi_2 \cdot \boldsymbol{\nabla}_s \zeta_1 - \boldsymbol{\nabla}_s \phi_1 \cdot \boldsymbol{\nabla}_s \zeta_2 + W_1 (\boldsymbol{\nabla}_s \zeta_1)^2.
\end{aligned}\right\} \quad \text{(A 6)}$$

Expanding ϕ_n around the reference surface at $\zeta = 0$,

$$\left.\begin{aligned}
\phi_1(\boldsymbol{x}, \zeta, t) &= \phi_1(\boldsymbol{x}, 0, t), \\
\phi_2(\boldsymbol{x}, \zeta, t) &= \phi_2(\boldsymbol{x}, 0, t) + \zeta_1 \kappa\phi_1(\boldsymbol{x}, 0, t), \\
\phi_3(\boldsymbol{x}, \zeta, t) &= \phi_3(\boldsymbol{x}, 0, t) + \zeta_1 \kappa\phi_2(\boldsymbol{x}, 0, t) + \zeta_2 \kappa\phi_1(\boldsymbol{x}, 0, t) + \tfrac{1}{2}\zeta_1^2 \kappa^2 \phi_1(\boldsymbol{x}, 0, t)
\end{aligned}\right\} \quad \text{(A 7)}$$

in (A 6), with some arrangement and cancellation of terms, gives the Hasselmann equations ((A 4) and (A 5)).

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